On the mathematical theory of splitting and Russian roulette techniques

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Splitting is an universal and potentially very powerful technique for increasing efficiency of simulation studies especially for the case of rare events evaluation. The idea of this technique goes back to von Neumann (see [Kan and Harris,1951]). A theory of multilevel splitting is rather well developed (see e.g.[L'Ecuryer et al.2006] and references therein). However, in the most of papers strongly restrictive assumptions on transition probabilities are imposed. It may be not very important if we are interested only in rare events but not acceptable if we like to evaluate behavior of a system in a more complete way. In [Ermakov and Melas, 1995] and [Melas, 1997] a more general theory is developed. It is based on the introduction of a probability measure governing the procedures of splitting and Russian roulette. This theory includes the problem in the frame of optimal experimental design. In the present talk we will describe basic ideas and results of this theory and give a few examples.

To illustrate the basic ideas let us consider a finite Markov chain with the state space $\Omega = \{0, 1, \ldots, m\}$, a transition matrix P and a stationary distribution $\pi = (\pi_0, \ldots, \pi_m)^T$. Assume that for an integer number N all elements of the matrix P^N are positive. The problem is to evaluate the vector π by simulation.

As the immediate simulation method we will consider the well known regenerative approach. Let $\{i^*\}$ be the initial state. The method consists of the simulation of several paths of the chain beginning in $\{i^*\}$ up to returning to this state. As the estimator of π_i $(i = 0, 1, \ldots, m)$ the number of hits to the state $\{i\}$ by all paths divided by the number of all steps by all paths is used.

Let $\beta_0, \ldots, \beta_m$ be arbitrary fixed nonnegative numbers and it could be $\beta_i = 0$ if the state $\{i\}$ is a state in which every paths will be cancelled.

When a path transits from a state i to a state j, $i \in \Omega$, $j \in \Omega \backslash \{i^*\}$ when $\beta_j \le \beta_i$ we will simulate additionally η paths beginning in the state j where

$$\eta = \begin{cases} \lfloor \beta_j/\beta_i \rfloor - 1 & \text{with a probability} \quad 1-\alpha \\ \lfloor \beta_j/\beta_i \rfloor & \text{with a probability} \quad \alpha, \end{cases}$$

$\alpha = \beta_j/\beta_i - \lfloor \beta_i/\beta_i \rfloor$ and $\lfloor a \rfloor$ designates the integer part of a. When $\beta_j < \beta_i$ we will cancel the current path with probability β_j/β_i. As the estimator of π_k ($k = 0, 1, \ldots, m$) we will take the number of hits into the state k by all path multiplied by β_k and divided by the sum of all such numbers. One of β_k could be equal to 0 since $\pi_k = 1 - \sum_{i \neq k} \pi_i$ and one of the π_i could be recalculated by others.

We will call experimental design the discrete probability measure $\tau = \{\tau_0, \ldots, \tau_m\}$, $\tau_i \geq 0$ ($i = 0, \ldots, m$), $\sum \tau_i = 1$, where

$$\tau_i = \beta_i \pi_i \Big/ \sum_{k=0}^{m} \beta_k \pi_k.$$

In [Ermakov and Melas, 1995] it was proved that the estimators are asymptotically unbiased and the covariance matrix of the estimators of $(\pi_1, \ldots, \pi_m)$ multiplied by the expected number of steps is approximately equal

$$D(\tau) = W^T B(\tau) W,\ W = (I - P_{\_})^{-1},$$

where $P_{\_}$ is the matrix P with the first row and the first column rejected,

$$B(\tau) = \sum_{k=0}^{m-1} \frac{\pi_k^2}{\tau_k} B_k,\ B_k = (p_{ki}\delta_{ij} - p_{ki}p_{kj})_{i,j=1}^{m},$$

$$\delta_{ij} = 1 \text{ for } i = j \text{ and } \delta_{ij} = 0 \text{ for } i \neq j.$$

The immediate simulation method is the particular case when $\beta_0 = \beta_1 = \ldots = \beta_m$ that is $\tau = \pi$. The natural efficiency criterion for the problem is the determinant of $D(\tau)$. Since W does not depend on τ the criterion is reduced to $\det B(\tau)$.

A design τ^* is called D-optimal design if it minimizes the magnitude $\det B(\tau)$. In a similar way we can consider other optimality criteria and more general types of chains. A few examples will be given in the talk.

References

Ermakov, S. M., and V. B. Melas. (1995). *Design and analysis of simulation experiments.* Dordrecht, London: Kluwer Academic Publishers.

Kahn, H., and T. E. Harris. (1951). Estimation of particle transmission by random sampling. *National Bureau os Standards Applied Mathematical Series* 12:27–30.

L'Ecuyer, P., Demers, V., and B.Tuffin (2006). Splitting for rare-event simulation.In: *Proceedings of the 2006 Winter Simulation Conference*, 137–148,California Monterey.

Melas, V. B. (1997). On the efficiency of the splitting and roulette approach for sensitivity analysis. In: *Proceedings of the 1997 Winter Simulation Conference*, 269–274. Pissataway, NJ: IEEE Press.